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13. ABSTRACT *(Maximum 200 words)*
The primary objective of this grant has been the study of algorithms for solving ill-conditioned matrix equations arising in control, filtering, and system theory. Much of our work has concentrated on matrix Riccati and Lyapunov equations which are absolutely fundamental to the field. We have made significant advances on a number of fronts in the numerical solution of large-scale and ill-conditioned Lyapunov, Sylvester, and Riccati equations. Substantial progress has been made in other areas as well, including a new family of algorithms based on matrix interpolation for frequency response and related problems, a number of key advances in numerical linear algebra, algorithms for infinite-dimensional systems, a new theory of small-sample statistical condition estimation, and software implementations of many of our algorithms. Our results have been reported in over thirty scholarly articles.

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Solving Ill-Conditioned Matrix Equations in Control

FINAL TECHNICAL REPORT

Air Force Office of Scientific Research

Grant No. AFOSR-91-0240

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1 FACTUAL DATA

This section contains a listing of publications, graduate students supported in whole or in part under the grant, and honors/awards received. With respect to publications, our research advances have been reported in over 30 scholarly articles, including 14 in the leading IEEE and SIAM journals. Major publications are listed below and the narrative to follow in Section 2 is keyed to this list. A summary of significant progress made under the auspices of this grant is documented in the narrative.

1.1 Publications Supported by this AFOSR Grant

[1] Papadopoulos, P.M., A.J. Laub, C.S. Kenney, P. Pandey, G. Ianculescu, and J. Ly, "Optimal Control Study for the Space Station Solar Dynamic Power Module," *Proc. 30th IEEE Conf. on Decision and Control*, Brighton, England; December 1991; pp. 2224–2229.

[2] Ghavimi, A., C. Kenney, and A.J. Laub, "Local Convergence Analysis of Conjugate Gradient Methods for Solving Algebraic Riccati Equations," *IEEE Trans. Aut. Contr.*, 37(1992), 1062–1067.

[3] Gahinet, P., and A.J. Laub, "Algebraic Riccati Equations and the Distance to the Nearest Uncontrollable Pair," *SIAM J. Contr. Opt.*, 30(1992), 765–786.

[4] Kenney, C., and A.J. Laub, "On Scaling Newton's Method for Polar Decomposition and the Matrix Sign Function," *SIAM J. Matrix Anal. Appl.*, 13(1992), 688–706.

[5] Gudmundsson, T., C. Kenney, and A.J. Laub, "Scaling of the Discrete-Time Algebraic Riccati Equation to Enhance Stability of the Schur Solution Method," *IEEE Trans. Aut. Contr.*, 37(1992), 513–518.

[6] Williams, T., and A.J. Laub, "Orthogonal Canonical Forms for Second-Order Systems," *IEEE Trans. Aut. Contr.*, 37(1992), 1050–1052.

[7] Kenney, C.S., A.J. Laub, and P.M. Papadopoulos, "Matrix-Sign Algorithms for Riccati Equations," *IMA Journal of Mathematical Control and Information*, 9(1992), 331–344 (see also Kenney, C.S., A.J. Laub, and P.M. Papadopoulos, "Matrix Sign Algorithms for Riccati Equations," *Proc. IMA Conf. on Control: Modelling, Computation, Information*, Manchester, England; pp. 1–10; September 1992).

[8] Cheng, J., G. Ianculescu, C.S. Kenney, A.J. Laub, J. Ly, and P.M. Papadopoulos, "Control-Structure Interaction Study for the Space Station Solar Dynamic Power Module," *Control Systems Magazine*, Vol. 12, pp. 4–13, October 1992.

[9] Pandey, P., C. Kenney, and A.J. Laub, "Solving the Algebraic Riccati Equation on Supercomputers," in *Recent Advances in Mathematical Theory of Systems, Control, Networks, and Signal Processing*, Vol. II, H. Kimura and S. Kodama (eds.), MITA Press, Tokyo, 1992, pp. 3–8.

[10] Erickson, M.A., R.S. Smith, and A.J. Laub, "Calculating Finite-Dimensional Approximations of Infinite-Dimensional Linear Systems," *Proc. American Control Conf.*, Chicago, IL; June 1992; pp. 157–161.

[11] Kenney, C., S. Stubberud, and A.J. Laub, "A Rational Interpolation Method to Compute Frequency Response," *Proc. Fifth Annual NASA/NSF/DOD Workshop on Aerospace Computational Control*, Santa Barbara, CA; August 1992; pp. 413–426.

[12] Erickson, M.A., A.J. Laub, and R.S. Smith, "Calculating Eigenvalues and Eigenfunctions of Hyperbolic Systems," *Proc. 31st IEEE Conf. on Decision and Control*, Tucson, Arizona; pp. 145–146; December 1992.

[13] Ianculescu, G.D., J. Ly, A.J. Laub, and P.M. Papadopoulos, "Space Station *Freedom* Solar Array H_∞ Control," *Proc. 31st IEEE Conf. on Decision and Control*, Tucson, Arizona; pp. 639–640; December 1992.

[14] Kenney, C.S., A.J. Laub, and S.C. Stubberud, "Frequency Response Computation Via Rational Interpolation," *IEEE Trans. Aut. Control*, 38(1993), 1203–1213 (see also Kenney, C.S., S. Stubberud, and A.J. Laub, "Frequency Response Computation Via Rational Interpolation," *Proc. 1992 IEEE Symposium on Computer-Aided Control System Design*, Napa, CA; March 1992; pp. 188–195).

[15] Kenney, C., A.J. Laub, and P. Papadopoulos, "A Newton-Squaring Algorithm for Computing the Negative Invariant Subspace of a Matrix," *IEEE Trans. Aut. Control*, 38(1993), 1284–1289.

[16] Pandey, P., and A.J. Laub, "Numerical Issues in Robust Control Design Techniques," in *Control and Dynamic Systems - Advances in Theory and Applications: Vol. 55, Digital and Numeric Techniques and Their Applications in Control Systems*, C.T. Leondes (ed.), Academic, San Diego, 1993, pp. 25–50.

[17] Stubberud, S.C., A.J. Laub, and C.S. Kenney, "Computation of Frequency Response of Descriptor Systems by Rational Interpolation," in *Control and Dynamic Systems - Advances in Theory and Applications: Vol. 56, Digital and Numeric Techniques and Their Applications in Control Systems*, C.T. Leondes (ed.), Academic, San Diego, 1993, pp. 267–301.

[18] Holthaus, M., C.S. Kenney, and A.J. Laub, "Numerical Methods for Studying Parameter Dependence of Solutions to Schrödinger's Equation," in *Differential Equations, Dynamical Systems, and Control Science (A Festschrift in Honor of Lawrence Markus)*," (K.D. Elworthy, W.N. Everitt, and E.B. Lee, eds.), Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, New York, 1993, pp. 101–114.

[19] Papadopoulos, P.M., C.S. Kenney, and A.J. Laub, "Least-Squares Solution of Ill-Conditioned Lyapunov Equations," *Proc. American Control Conf.*, San Francisco, California; June 1993; pp. 1588–1592.

[20] Hench, J.J., and A.J. Laub, "On the Numerical Solution of the Discrete-Time Periodic Riccati Equation," *Proc. 11th Int'l. Symp. on Math. Theory of Networks and Systems (MTNS)*; Regensburg, Germany; August 1993.

[21] Pandey, P., and A.J. Laub, "A Note on Invariant Subspaces of Hamiltonian Matrices," *Proc. 32nd IEEE Conf. on Decision and Control*, San Antonio, Texas; December 1993; pp. 3150–3155.

[22] Kenney, C.S., and A.J. Laub, "A Statistical Approach to Condition Estimation," *Proc. 32nd IEEE Conf. on Decision and Control*, San Antonio, Texas; December 1993; pp. 3156–3161.

[23] Kenney, C., and A.J. Laub, "Small-Sample Statistical Condition Estimates for General Matrix Functions," *SIAM J. Sci. Comp.*, 15(1994), 36–61.

[24] Patel, R.V., A.J. Laub, and P.M. Van Dooren, "Introduction and Survey," Part 1 of *Numerical Linear Algebra Techniques for Systems and Control*, R.V. Patel, A.J. Laub, and P.M. Van Dooren (eds.), IEEE Press, Piscataway, New Jersey, 1994, pp. 1–35.

[25] Hench, J.J., C.S. Kenney, and A.J. Laub, "Methods for the Numerical Integration of Hamiltonian Systems," to appear in *Circuits, Systems, and Signal Processing*, 1994.

[26] Hench, J.J., and A.J. Laub, "Numerical Solution of the Discrete-Time Periodic Riccati Equation," to appear in *IEEE Trans. Aut. Control*, June 1994.

[27] Gudmundsson, T., and A.J. Laub, "Approximate Solution of Large Sparse Lyapunov Equations," to appear in *IEEE Trans. Aut. Control*, May 1994 (see also Gudmundsson, T., and A.J. Laub, "Computing the Hankel Singular Values of Large, Sparse Linear Systems," *Proc. American Control Conf.*, Chicago, IL; June 1992; pp. 664–665).

[28] Kenney, C.S., and A.J. Laub, "A Hyperbolic Tangent Identity and the Geometry of Padé Sign Function Iterations," to appear in *Numerical Algorithms*, 1994.

[29] Erickson, M.A., R.S. Smith, and A.J. Laub, "Finite-Dimensional Approximation and Error Bounds for Spectral Systems with Partially Known Eigenstructure," to appear in *IEEE Trans. Aut. Control*, 1994 (see also *Proc. 1993 CDC*, pp. 1848–1853).

[30] Ghavimi, A.R., and A.J. Laub, "An Implicit Deflation Method for Ill-Conditioned Sylvester and Lyapunov Equations," to appear in *Int. J. Control*, 1994.

[31] Erickson, M.A., R.S. Smith, and A.J. Laub, "Power Methods for Calculating Eigenvalues and Eigenfunctions of Spectral Operators on Hilbert Spaces," to appear in *Int. J. Control*, 1994.

[32] Ghavimi, A.R., and A.J. Laub, "Computation of Approximate Null Vectors of Sylvester and Lyapunov Operators," to appear in *IEEE Trans. Aut. Control*, 1995.

[33] Gudmundsson, T., C. Kenney, and A.J. Laub, "Small-Sample Statistical Estimates for Matrix Norms," to appear in *SIAM J. Matr. Anal. Appl.*, 1995.

[34] Erickson, M.A., and A.J. Laub, "An Algorithmic Test for Checking Stability of Feedback Spectral Systems," to appear in *Automatica*, 1995.

[35] Ghavimi, A.R., and A.J. Laub, "Backward Error, Sensitivity, and Refinement of Computed Solutions of Algebraic Riccati Equations," to appear in *Numerical Linear Algebra with Applications*, 1995.

1.2 Graduate Students Supported by this AFOSR Grant

The following six Ph.D. students have been supported in whole or in part by this AFOSR grant:

1. Stephen C. Stubberud (September 1992)
   Fast and Reliable Approximation Methods for Matrix Problems in Control

2. John J. Hench (September 1992)
   Numerical Methods for Periodic Linear Systems

3. Thorkell T. Gudmundsson (September 1992)
   Implicit Matrix Approximations in Control Theory

4. Philip M. Papadopoulos (June 1993)
   Numerical Algorithms for Large-Scale and Ill-Conditioned Matrix Equations in Control

5. Ali R. Ghavimi (June 1993)
   Iterative Methods for Large-Scale and Nearly Singular Matrix Equations in Control Theory

6. Mark A. Erickson (expected, June 1994)
   Computational Methods for Infinite-Dimensional Control Systems

Note that 5 of the above 6 students are U.S. citizens. Gudmundsson is a citizen of Iceland. In addition to the above, two other Ph.D. students also work in the P.I.'s research group: Thomas A. Bryan (October 1992), *Analysis and Sensitivity of Direction-Finding Algorithms* and Michael Reese (Ph.D. expected, June 1996). Both are U.S. citizens.

1.3 External Honors, Awards, etc.

The following external recognition has been accorded the P.I. during the course of this grant:

1. In 1991, the P.I. was the (elected) President of the IEEE Control Systems Society. In the course of this service, he authored many editorials and "President's Messages" relating to the field of control. These appear in *IEEE Trans. Aut. Contr.*, 36(1991), p. 2 and in *IEEE Control Systems Magazine*, 1991: January (p. 107), February (p. 50), April (p. 79), June (pp. 48–49), August (p. 3), October (p. 36), December (p. 9).

2. In December 1991, the P.I. was presented with a Distinguished Member Award by the IEEE Control Systems Society, one of about 40 so recognized in this 11,000-member professional society.

3. In December 1993, the P.I. received the *Control Systems Technology Award* from the IEEE Control Systems Society "for pioneering efforts and continuously advancing the state of the art in Computer-Aided Control System Design." This is a major career research award.

2 NARRATIVE

Our research has the following basic goals:

- Create new algorithms for important generic numerical problems arising in control engineering (and other engineering and scientific fields), especially those involving large-scale and ill-conditioned matrix equations.

- Facilitate the development of next-generation (parallel and vector) algorithms and codes for computer-aided control system design.

- Raise the level of computational capability and understanding in the academic, industrial, and governmental communities [24].

The primary objective of this particular grant has been the study of algorithms for solving ill-conditioned matrix equations arising in control, filtering, and system theory. Much of our work has concentrated on matrix Riccati and Lyapunov equations which are absolutely fundamental to the field. Substantial progress has been made in other areas as well and we give the highlights of some of the more exciting contributions below.

1. Large-Scale and Ill-Conditioned Matrix Equations: We have made substantial progress in pursuing various iterative algorithms for solving large-scale and ill-conditioned computing problems in control. For example, because of our previous and continuing research, the matrix sign method has now become a standard tool for solving large-scale and ill-conditioned matrix Riccati equations, as arise, for example, in H_∞ control and distributed parameter control systems [9]. An extensive survey of matrix-sign-function properties, algorithms, and applications has been published in one of the leading British applied mathematics journals [7]. Our new algorithms have proven especially amenable to implementation on both parallel and vector computers, and Riccati equations of order, say, 100 can be solved reliably in less than a second! Moreover, our research is helping establish the

algorithmic foundation for some of the next generation of computer-aided control design software. We have applied our new algorithms successfully to some "real-world" large-scale problems. In [1] and [8] a discussion is given of the numerical solution of Riccati equations of order 556 (involving Hamiltonian matrices of order 1112) in joint work with Rockwell's Rocketdyne Division. The problem derives from a model associated with Space Station Freedom in which 278 modes are included [13].

We are continuing a rather substantial research effort on the matrix sign function and its applications, including additional improvements to the basic Newton iteration such as "inverse-free" methods [15] and better understanding of the use of scaling factors to accelerate convergence [4]. Other numerical research, crucial to successful implementation as software, is devoted to issues of conditioning, numerical stability, iterative improvement, and deeper understanding of the geometry of convergence including global convergence. The latter has been greatly simplified by the discovery of a beautiful new unifying theory based on a formula involving hyperbolic tangents [28].

In addition to matrix sign iterations, we have made significant progress in studying other iterative algorithms for solving large-scale and ill-conditioned computing problems in control. Among these are conjugate gradient methods for Riccati and other general matrix equations [2], least-squares solution methods for nearly singular Lyapunov and Sylvester equations [19], implicit and explicit deflation methods via inverse iteration for certain nearly singular matrix equations [30], and new methods for the computation of approximate null vectors of Sylvester and Lyapunov operators [32]. We have also developed an efficient algorithm for estimating the dominant eigenvalues and corresponding eigenvectors of the solution to a Lyapunov equation without first solving the equation explicitly [27]. Such an approach is necessary when coefficient matrices are large but sparse. An immediate application of this method is to balanced-truncation order reduction of linear systems. Curiously, exisiting methods for balancing are not actually applicable to large sparse systems.

2. Matrix Interpolation for Frequency Response and Related Problems: The P.I.'s 1981 Hessenberg algorithm is one of the most efficient and reliable algorithms for computing a frequency response matrix from state-space data (including so-called descriptor models). We have recently developed a novel algorithm, based on rational interpolation of matrix-valued functions, for enhancing the Hessenberg method [11], [14], [17]. A rather clever use of a certain resolvent identity avoids the potential inaccuracies inherent in the subtraction of nearly equal values in the calculation of finite differences. When coupled with a pole/zero cancellation method, the resulting interpolation algorithm is accurate and efficient. Somewhat serendipitously, the error in this procedure has the form of a modified frequency response matrix, which means that the interpolation algorithm can be used to approximate both the response matrix and the error. Commercial software houses are anxious to implement this algorithm in MATLAB and XMATH to support their next-generation algorithmic capability. Other applications of this interpolation method include the evaluation of matrix exponentials and a parameter dependence study of solutions to the periodically time-dependent Schrödinger equation as approximated by a finite system of ordinary differential equations [18]. We have been able to demonstrate a dramatic decrease in computation time for an example of the latter used in modeling an electron confined to a quantum well.

3. Advances in Numerical Linear Algebra: As a by-product of our control-related research, we study and develop many new algorithms of independent interest in the field of numerical linear algebra. For example, a significant new algorithm has recently been discovered in connection with the problem of computing the periodic nonnegative definite stabilizing solution of the discrete-time periodic Riccati equation [20], [26]. The algorithm determines a simultaneous triangularization, by

orthogonal equivalences, of a sequence of matrices associated with a cyclic pencil formulation related to the Euler-Lagrange equations, and no matrix products need be formed explicitly. Algorithms for the continuous-time periodic Riccati equation have also been studied [25].

A key question in any serious numerical computing is to ask how near a given problem is to a problem possessing some sort of undesirable behavior such as singularity or instability. A significant paper [3] has been published in this area of matrix "nearness" problems. A thorough mathematical treatment is given in [3] for the problem of determining the nearness to uncontrollability of a given controllable state-space model. The key tool used in the analysis is a connection between nearness to unstabilizability and the behavior of the unique symmetric positive definite stabilizing solution of an associated algebraic Riccati equation. We have also continued to make noteworthy progress in other aspects of the numerical solution of Riccati equations. For example, in [5] we have succeeded in deriving and extending the scaling results of Kenney, Laub, and Wette (*Sys. Contr. Lett.*, 12(1989), 241-150) for the Schur method to the discrete-time case. Both theoretical and computable bounds are determined and we note that the discrete-time case turned out to be somewhat nontrivial to handle.

An important numerical result relating to the matrix triples commonly found in so-called matrix second-order models has been published in [6]. The basic idea is to establish which canonical forms are obtainable under orthogonal equivalence for the standard matrix triple consisting of a mass matrix, a stiffness matrix, and a damping matrix. Equivalence under orthogonal transformations is, of course, crucial for numerical reliability. It is established that an arbitrary damping model can not be used but that orthogonal reduction of the commonly used modal damping model can be so reduced.

Another significant advance in numerical linear algebra is a complete new backward error analysis for Lyapunov, Sylvester, and Riccati equations [35]. This theory has immediate application to sensitivity analysis and accuracy assessment of these key matrix equations in control.

Finally, another key result with a strong numerical linear algebra flavor has been made in the robust control area. A new procedure has been developed that obviates the need for explicit Riccati equation solution in the standard two-Riccati-equation state-space approach to the H_∞ problem. Instead, our new method works directly and only with bases for invariant subspaces [16], [21].

4. Algorithms for Infinite-Dimensional Systems: Control system analysis and synthesis problems associated with linear time-invariant infinite-dimensional systems have received much attention in recent years. Some of our recent research focuses on the use of the eigenstructure of certain operators on infinite-dimensional Hilbert spaces to compute answers to control problems. In particular, our results can be applied to systems that do not have eigenvalues and eigenvectors available in closed form, e.g., systems described by partial differential equations (PDEs) with spatially variant parameters, systems with two- or three-dimensional domains with complicated boundary shapes, and so forth.

Three basic problems are solved for classes of systems that can be formulated as bounded spectral systems. First, power, inverse power, and orthogonal iteration methods are formulated for directly calculating eigenvalues and eigenfunctions of classes of spectral operators associated with the systems of interest [12], [31]. Second, bounds are derived on the error incurred by approximating canonical parabolic and hyperbolic systems with finite-dimensional modal models [10], [29]. These bounds require only a finite number of eigenvalues and eigenfunctions, which can be calculated with the proposed power methods. Next, a computable test is formulated for verifying the stability or instability of the feedback connection of a class of spectral systems and either state feedback or a finite-dimensional linear time-invariant controller [34]. This test also requires only a finite number of eigenvalues and eigenfunctions of the spectral system in question.

Finally, a collection of MATLAB functions has been developed that implements the power methods, frequency-domain model and bound calculations, and stability tests developed and studied in our research. The package is designed to be used in conjunction with the μ-Tools toolbox to enable the user to perform modeling, H_∞ and D-K iteration controller synthesis, stability analysis, simulation, and animated visualization for systems described by parabolic and hyperbolic PDEs, possibly with spatially variant parameters. The utility of the collection has been demonstrated on several detailed examples.

5. Small-Sample Statistical Condition Estimation: A major and fundamental new theory has been developed in the area of statistical estimation of condition [22], [23]. This is the principal focus of our new AFOSR grant.

Efficient estimation of matrix norms has long been a central problem in condition theory, especially for situations where the matrix in question is not known explicitly or is too expensive to compute directly. Generally speaking, the best that one can hope for is that the result of applying a matrix to a given vector is available. In a more general setting, we may wish to measure the sensitivity of a function that maps matrices into matrices by estimating the norm of the linearization (Fréchet derivative) of the function about a particular matrix. This can be approximated in a very useful and efficient way using finite differences. Sometimes, and this is generally true of so-called power methods for computing large singular values, it is also necessary that the transpose of a matrix, or its application to a given vector, also be available. This is not always possible in an efficient way in many important applications, particularly when the domain and co-domain of a function have different dimensions.

Because of this problem, we have developed an exciting new form of condition estimation, wherein the transpose requirement is dropped and it is assumed only that matrix-vector products can be obtained at a reasonable cost. Using a finite-difference approach, this cost is generally no more than the cost of one extra function evaluation, and for many problems can be less.

Somewhat surprisingly, the statistical theory associated with the norms of matrix-vector products for random vectors can be worked out in great detail and a rather complete analysis derived that predicts the accuracy of norm estimates for a matrix from just a few matrix-vector products. The theory is based on the distribution of inner products between a fixed vector $v \in \mathbb{R}^n$ and certain randomly selected unit vectors z, and many useful analytical results are available. By taking more than one inner product, say m of them, we obtain an mth-order estimate for the norm of v. That is, the probability of a bad estimate (off by more than a factor ω) is less than a constant divided by ω^m. Thus, only a few inner products are needed to render the possibility of a bad estimate for the norm of v extremely small.

This procedure can be extended to estimate the Frobenius norm of a matrix M with just a few matrix-vector products Mz_i, although this case is, in many respects, much more difficult. Some important first results and a key conjecture have been reported in [33].

Our statistical condition estimation theory is applicable to a wide variety of applications, many of which are of interest to control engineers, including sensitivity of the matrix exponential, sensitivity of Lyapunov and Riccati equations, distance to uncontrollability, and so forth. We are currently investigating many of these problems in our ongoing research.

6. Software Implementations: One of the most important things we can do in our research is to implement our algorithms as efficient, robust, and widely available software. Experience has shown that if we don't personally undertake that critical final step, others can't or won't. We have recently created new Riccati codes to run under MATLAB. Not only are these the most efficient currently available, but also they handle most "singular" cases, including Riccati equations of

the type that arise in both continuous-time and discrete-time H_∞ control. Some of our recent perturbation results for the Hamiltonian eigenvalue problem will be useful here. The new Riccati codes will be very useful in spectral factorization and a wide variety of other applications. They will also be exploited heavily in a new toolbox project with researchers at INRIA. This toolbox, called LMI-Lab, is designed to handle linear matrix inequalities, a topic of intense current interest.

Other software projects in which we are currently engaged include more efficient adaptive codes for frequency response and implementation of our recent results on small-sample statistical condition estimation for matrix-valued functions and equations. These will be key features of next-generation algorithmic capability in CACSD.

Finally, in addition to MATLAB-type software implementations, we are also looking at real-time implementations and implementations of appropriate algorithms on massively parallel machines.

In summary, we continue to be excited about the progress that has been made on a wide variety of important numerical problems arising in control and system theory. We are extremely enthusiastic about the prospects and opportunities for further research.